

# Cayley Automatic Groups and Numerical Characteristics of Turing Transducers

Dmitry Berdinsky

Department of Computer Science, The University of Auckland,  
Private Bag 92019, Auckland 1142, New Zealand  
berdinsky@gmail.com

**Abstract.** This paper is devoted to the problem of finding characterizations for Cayley automatic groups. The concept of Cayley automatic groups was recently introduced by Kharlampovich, Khoussainov and Miasnikov. We address this problem by introducing three numerical characteristics of Turing transducers: growth functions, Følner functions and average length growth functions. These three numerical characteristics are the analogs of growth functions, Følner functions and drifts of simple random walks for Cayley graphs of groups. We study these numerical characteristics for Turing transducers obtained from automatic presentations of labeled directed graphs.

**Keywords:** Cayley automatic groups, Turing transducers, growth function, Følner function, random walk.

## 1 Introduction

This paper contributes to the field of automatic structures [11,12,13] with particular emphasis on Cayley automatic groups [10]. Recall that a finitely generated group  $G$  is called Cayley automatic if for some set of generators  $S$  the labeled directed Cayley graph  $\Gamma(G, S)$  is an automatic structure (or, FA-presentable). All automatic groups in the sense of Thurston are Cayley automatic. However, the class of Cayley automatic groups is considerably wider than the class of automatic groups. For example, all finitely generated nilpotent groups of nilpotency class at most two and all fundamental groups of three-dimensional manifolds are Cayley automatic [10]. The Baumslag-Solitar groups are Cayley automatic [1]. Cayley automatic groups retain the key algorithmic properties which hold for automatic groups: the word problem for Cayley automatic groups is decidable in quadratic time, the conjugacy problem for Cayley biautomatic groups is decidable, and the first order theory for Cayley graphs of Cayley automatic groups is decidable.

Oliver and Thomas found a characterization of FA-presentable groups by showing that a finitely generated group is FA-presentable if and only if it is virtually abelian [16]. Their result is based partly on the celebrated Gromov's theorem on groups of polynomial growth. But, the problem of finding characterizations for Cayley automatic groups is more complicated, and it seems to

require new approaches. In this paper we address this problem by introducing some numerical characteristics for Turing transducers of the special class  $\mathcal{T}$ .

In Section 2 we define the class of Turing transducers  $\mathcal{T}$ . Then we show that automatic presentations of Cayley graphs of groups can be expressed in terms of Turing transducers of the class  $\mathcal{T}$ . This explains why study of admissible asymptotic behavior for some numerical characteristics of Turing transducers of the class  $\mathcal{T}$  is relevant to the problem of finding characterizations for Cayley automatic groups. In Section 3 we introduce three numerical characteristics for Turing transducers of the class  $\mathcal{T}$ . In this paper, wreath products of groups are used as the source of examples of Cayley automatic groups. Therefore, in Section 4 we briefly recall basic definitions for wreath products of groups. In Section 5 we discuss asymptotic behavior of the numerical characteristics of Turing transducers of the class  $\mathcal{T}$ . Section 6 concludes the paper.

## 2 Turing Transducers of the Class $\mathcal{T}$ and Automatic Presentations of Labeled Directed Graphs

Recall that a  $(k+1)$ -tape Turing transducer  $T$  for  $k \geq 1$  is a multi-tape Turing machine which has one input tape and  $k$  output tapes. See, e.g., [14, § 10] for the definition of Turing transducers. The special class of Turing transducers  $\mathcal{T}$  that we consider in this paper is described as follows. Let us be given a  $(k+1)$ -tape Turing transducer  $T \in \mathcal{T}$  and an input word  $x \in \Sigma^*$ . Initially, the input word  $x$  appears on the input tape, the output tapes are completely blank and all heads are over the leftmost cells. First the heads of  $T$  move synchronously from the left to the right until the end of the input  $x$ . Then the heads make a finite number of steps (probably no steps) further to the right, where this number of steps is bounded from above by some constant which depends on  $T$ . After that, the heads of  $T$  move synchronously from the right to the left until it enters a final state with all heads over the leftmost cells.

We say that  $T$  accepts  $x$  if  $T$  enters an accepting state; otherwise,  $T$  rejects  $x$ . Let  $L \subseteq \Sigma^*$  be the set of inputs accepted by  $T$ . We say that  $T$  translates  $x \in L$  into the outputs  $y_1, \dots, y_k$  if for the word  $x$  fed to  $T$  as an input,  $T$  returns the word  $y_i$  on the  $i$ th output tape of  $T$  for every  $i = 1, \dots, k$ . It is assumed that for every input  $x \in L$ , the output  $y_i \in L$  for every  $i = 1, \dots, k$ . Let  $L' \subseteq L^k$  be the set of all  $k$ -tuples of outputs  $(y_1, \dots, y_k)$ . We say that  $T$  translates  $L$  into  $L'$ .

For a given finite alphabet  $\Sigma$  put  $\Sigma_\diamond = \Sigma \cup \{\diamond\}$ , where  $\diamond \notin \Sigma$ . The convolution of  $n$  words  $w_1, \dots, w_n \in \Sigma^*$  is the string  $w_1 \otimes \dots \otimes w_n$  of length  $\max\{|w_1|, \dots, |w_n|\}$  over the alphabet  $\Sigma_\diamond^n$  defined as follows. The  $k$ th symbol of the string is  $(\sigma_1, \dots, \sigma_n)$ , where  $\sigma_i, i = 1, \dots, n$  is the  $k$ th symbol of  $w_i$  if  $k \leq |w_i|$  and  $\diamond$  otherwise. The convolution  $\otimes R$  of a  $n$ -ary relation  $R \subseteq \Sigma^{*n}$  is defined as  $\otimes R = \{w_1 \otimes \dots \otimes w_n \mid (w_1, \dots, w_n) \in R\}$ . Recall that a  $n$ -tape synchronous finite automaton is a finite automaton over the alphabet  $\Sigma_\diamond^n \setminus \{(\diamond, \dots, \diamond)\}$ . Let  $T \in \mathcal{T}$ . Lemma 1 below shows connection between Turing transducers of the class  $\mathcal{T}$  and multi-tape synchronous finite automata.

**Lemma 1.** *There exists a  $(k + 1)$ -tape synchronous finite automaton  $\mathcal{M}$  such that a convolution  $x \otimes y_1 \otimes \cdots \otimes y_k \in \Sigma_\diamond^{(k+1)*}$  is accepted by  $\mathcal{M}$  iff  $T$  translates the input  $x$  into the outputs  $y_1, \dots, y_k$ . In particular, the language  $L$  is regular.*

*Proof.* The lemma can be obtained straightforwardly from the following two well known facts. The first fact is that the class of regular languages is closed under reverse. The second fact is as follows. Let the convolutions  $\otimes R_1$  and  $\otimes R_2$  of two relations  $R_1 = \{(x, y) | x, y \in \Sigma^*\}$  and  $R_2 = \{(y, z) | y, z \in \Sigma^*\}$  be accepted by two-tape synchronous finite automata. Then the convolution  $\otimes R$  of the relation  $R = \{(x, z) | \exists y[(x, y) \in R_1 \wedge (y, z) \in R_2]\}$  is accepted by a two-tape synchronous finite automaton.  $\square$

In other words, one can say that multi-tape synchronous finite automata simulate Turing transducers of the class  $\mathcal{T}$ . In different context, the notion of simulation for finite automata appeared, e.g., in [3, 4].

For a given  $k$ , put  $\Sigma_k = \{1, \dots, k\}$ . Let  $T \in \mathcal{T}$  be a  $(k + 1)$ -tape Turing transducer translating a language  $L$  into  $L' \subseteq L^k$ . We construct the labeled directed graph  $\Gamma_T$  with the labels from  $\Sigma_k$  as follows. The set of vertices  $V(\Gamma_T)$  is identified with  $L$ . For given  $u, v \in L$  there is an oriented edge  $(u, v)$  labeled by  $j \in \Sigma_k$  if  $T$  translates  $u$  into some outputs  $w_1, \dots, w_k$  such that  $w_j = v$ . It is easy to see that each vertex of the graph  $\Gamma_T$  has  $k$  outgoing edges labeled by  $1, \dots, k$ .

Let  $\Gamma$  be a labeled directed graph for which every vertex has  $k$  outgoing edges labeled by  $1, \dots, k$ . Recall that  $\Gamma$  is called automatic if there exists a bijection between a regular language and the set of vertices  $V(\Gamma)$  such that for every  $j \in \Sigma_k$  the set of oriented edges labeled by  $j$  is accepted by a synchronous two-tape finite automaton. From Lemma 1 we obtain that  $\Gamma_T$  is automatic. Suppose that  $\Gamma$  is automatic. Lemma 2 below shows that  $\Gamma$  can be obtained as  $\Gamma_T$  for some  $(k + 1)$ -tape Turing transducer  $T \in \mathcal{T}$ .

**Lemma 2.** *There exists a  $(k + 1)$ -tape Turing transducer  $T \in \mathcal{T}$  for which  $\Gamma_T \cong \Gamma$ .*

*Proof.* The lemma can be obtained from the following fact. Let  $R = \{(x, y) | x, y \in L\}$  be a binary relation such that  $\otimes R$  is recognized by a two-tape synchronous finite automaton, where  $L$  is a regular language. Suppose that for every  $x \in L$  there exists exactly one  $y \in L$  such that  $(x, y) \in R$ . Then there exists a two-tape Turing transducer  $T_R \in \mathcal{T}$  for which  $T_R$  translates  $x$  into  $y$  iff  $(x, y) \in R$  and  $T_R$  rejects  $x$  iff  $x \notin L$ . The construction of the Turing transducer  $T_R$  can be found, e.g., in [6, Theorem 2.3.10]. The resulting  $(k + 1)$ -tape Turing transducer  $T \in \mathcal{T}$  is obtained as the combination of  $k$  two-tape Turing transducers  $T_{R_1}, \dots, T_{R_k}$ , where  $R_1, \dots, R_k$  are the binary relations defined by the directed edges of  $\Gamma$  labeled by  $1, \dots, k$ , respectively.  $\square$

Lemmas 1 and 2 together imply the following theorem.

**Theorem 3.** *The labeled directed graph  $\Gamma$  is automatic iff there exists a Turing transducer  $T \in \mathcal{T}$  for which  $\Gamma \cong \Gamma_T$ .  $\square$*

Let  $\Gamma(G, S)$  be a Cayley graph for some set of generators  $S = \{s_1, \dots, s_k\}$ . Let us fix an order of elements in  $S$  as  $s_1, \dots, s_k$ . We say that the Cayley graph  $\Gamma(G, S)$  is presented by  $T \in \mathcal{T}$  if, after changing labels from  $j$  to  $s_j$  for every  $j \in \Sigma_k$  in  $\Gamma_T$ ,  $\Gamma_T \cong \Gamma(G, S)$ . The isomorphism  $\Gamma_T \cong \Gamma(G, S)$  defines the bijection  $\psi : L \rightarrow G$  up to the choice of the word of  $L$  corresponding to the identity  $e \in G$ . By Theorem 3 we obtain that if  $\Gamma(G, S)$  is presented by  $T \in \mathcal{T}$ , then  $G$  is a Cayley automatic group and  $T$  provides an automatic presentation for the Cayley graph  $\Gamma(G, S)$ . Moreover, for each automatic presentation of  $\Gamma(G, S)$  there is a corresponding Turing transducer  $T \in \mathcal{T}$  for which  $\Gamma(G, S)$  is presented by  $T$ .

### 3 Numerical Characteristics of Turing Transducers

We now introduce three numerical characteristics for Turing transducers of the class  $\mathcal{T}$ . Let  $T \in \mathcal{T}$  be a  $(k+1)$ -tape Turing transducer translating a language  $L$  into  $L' \subseteq L^k$ . Given a word  $w \in L$ , feed  $w$  to  $T$ . Let  $w_1, \dots, w_k \in L$  be the outputs of  $T$  for  $w$ . We denote by  $T(w)$  the set  $T(w) = \{w_1, \dots, w_k\}$ . Given a set  $W \subseteq L$ , we denote by  $T(W)$  the set  $T(W) = \bigcup_{w \in W} T(w)$ . Let us choose a word  $w_0 \in L$ . Put  $W_0 = \{w_0\}$ ,  $W_1 = T(W_0)$  and, for  $i > 1$ , put  $W_{i+1} = T(W_i)$ . Let  $V_n = \bigcup_{i=0}^n W_i$ ,  $n \geq 0$ . Put  $b_n = \#V_n$ .

- We call the sequence  $b_n, n = 0, \dots, \infty$  the growth function of the pair  $(T, w_0)$ .

For a given finite set  $W \subseteq L$  put

$$\partial W = \{w \in W \mid T(w) \not\subseteq W\}.$$

In other words,  $\partial W$  is the set of words  $w \in W$  for which at least one of the outputs of  $T$  for  $w$  is not in  $W$ . Define the function  $F\phi l(\varepsilon) : (0, 1) \rightarrow \mathbb{N}$  as

$$F\phi l(\varepsilon) = \min\{\#W \mid \#\partial W < \varepsilon \#W\}.$$

It is assumed that the function  $F\phi l(\varepsilon)$  is defined on the whole interval  $(0, 1)$ , i.e., for every  $\varepsilon \in (0, 1)$  the set  $\{W \mid \#\partial W < \varepsilon \#W\}$  is not empty.

- We call the sequence  $f_n = F\phi l(\frac{1}{n}), n = 1, \dots, \infty$  the Følner function of  $T$ .

Let  $M$  be a finite multiset consisting of some words of  $L$ . We denote by  $T(M)$  the multiset obtained as follows. Initially,  $T(M)$  is empty. Then, for every word  $w$  in  $M$  add the outputs of  $T$  for  $w$  to  $T(M)$ . If  $w$  has the multiplicity  $m$  in  $M$ , then this procedure must be repeated  $m$  times. Let  $M_0$  be the multiset consisting of the word  $w_0$  with the multiplicity one. Put  $M_1 = T(M_0)$  and, for  $i > 1$ , put  $M_{i+1} = T(M_i)$ . The total number of elements (multiplicities are taken into account) in the multiset  $M_n$  is  $k^n$ . Put  $\ell_n$  to be

$$\ell_n = \frac{\sum_{w \in M_n} m_w |w|}{k^n}, \quad (1)$$

where  $m_w$  is the multiplicity of a word  $w$  in  $M_n$  and  $|w|$  is the length of  $w$ . In other words,  $\ell_n$  is the average length of the words in the multiset  $M_n$ .

- We call the sequence  $\ell_n, n = 1, \dots, \infty$  the average length growth function of the pair  $(T, w_0)$ .

## 4 Wreath Products of Groups: Basic Notation

Most of the labeled directed graphs in this paper are obtained as Cayley graphs of wreath products of groups. For the sake of convenience we describe basic notation for restricted wreath products  $A \wr B$  in the present section. For more details on wreath products see, e.g., [9]. For given two groups  $A$  and  $B$ , we denote by  $A^{(B)}$  the set of all functions  $f : B \rightarrow A$  having finite supports. Recall that a function  $f : B \rightarrow A$  has finite support if the set  $\{x \in B \mid f(x) \neq e\}$  is finite, where  $e$  is the identity of  $A$ . Given  $f \in A^{(B)}$  and  $b \in B$ , we define  $f^b \in A^{(B)}$  as follows. Put  $f^b(x) = f(bx)$  for all  $x \in B$ . The group  $A \wr B$  is the set product  $A^{(B)} \times B$  with the group multiplication given by  $(f, b) \cdot (f', b') = (ff'^{b^{-1}}, bb')$ .

We denote by  $i_A$  the embedding  $i_A : A \rightarrow A \wr B$  for which  $i_A : a \mapsto (f_a, e)$ , where  $e$  is the identity of the group  $B$  and  $f_a \in A^{(B)}$  is the function  $f_a : B \rightarrow A$  such that  $f_a(e) = a$  and  $f_a(x)$  is the identity of the group  $A$  for every  $x \neq e$ . We denote by  $i_B$  the embedding  $i_B : B \rightarrow A \wr B$  for which  $i_B : b \mapsto (\mathbf{e}, b)$ , where  $\mathbf{e}$  is the identity of the group  $A^{(B)}$ ; in other words,  $\mathbf{e}$  is the function which maps all elements of  $B$  to the identity of the group  $A$ . For the sake of convenience we will identify  $A$  and  $B$  with the subgroups  $i_A(A) \leq A \wr B$  and  $i_B(B) \leq A \wr B$ , respectively. Let  $S_A = \{a_1, \dots, a_n\} \subseteq A$  and  $S_B = \{b_1, \dots, b_m\} \subseteq B$  be some sets of generators of the groups  $A$  and  $B$ , respectively. Then the set  $S = i_A(S_A) \cup i_B(S_B)$  is a set of generators of  $A \wr B$ . The Cayley graph  $\Gamma(A \wr B, S)$  can be obtained as follows. The vertices of  $\Gamma(A \wr B, S)$  are the elements of  $A \wr B$ , i.e., all pairs  $(f, b)$  such that  $f \in A^{(B)}$  and  $b \in B$ . The right multiplication of an element  $(f, b)$  by  $a_i, i = 1, \dots, n$  is  $(f, b)a_i = (\hat{f}, b)$ , where  $\hat{f}(s) = f(s)$  if  $s \neq b$  and  $\hat{f}(b) = f(b)a_i$ . The right multiplication of an element  $(f, b)$  by  $b_j, j = 1, \dots, m$  is  $(f, b)b_j = (f, bb_j)$ .

## 5 Asymptotic Behavior of the Numerical Characteristics

In this section we discuss asymptotic behavior of the numerical characteristics of Turing transducers of the class  $\mathcal{T}$ .

### 5.1 Growth Functions and Følner Functions

We first consider the behavior of growth function  $b_n, n = 0, \dots, \infty$  for Turing transducers of the class  $\mathcal{T}$ .

Let  $G$  be a group with a finite set of generators  $Q \subseteq G$ . Put  $S = Q \cup Q^{-1}$ . Recall that the growth function of the pair  $(G, Q)$  is the function  $\#B_n, n =$

$0, \dots, \infty$ , where  $\#B_n$  is the number of elements in the ball  $B_n = \{g \in G \mid \ell_S(g) \leq n\}$ . Let  $T \in \mathcal{T}$  be a Turing transducer translating a language  $L$  into  $L' \subseteq L^k$ , where  $k = \#S$ . Choose any word  $w_0 \in L$ . The following claim is straightforward.

*Claim.* Suppose that the Cayley graph  $\Gamma(G, S)$  is presented by  $T$ . Then the growth function  $b_n$  of the pair  $(T, w_0)$  coincides with the growth function of the pair  $(G, Q)$ .  $\square$

One of the important questions in the group theory is whether or not for a given pair  $(G, Q)$  the growth series is rational. A similar question naturally arises for a pair  $(T, w_0)$ . It is easy to show an example of a pair  $(T, w_0), T \in \mathcal{T}$  for which the growth series is not rational.

*Example 4.* Stoll proved that the growth series of the Heisenberg group  $H_5$  with respect to the standard set of generators is not rational [18]. The Cayley graph of  $H_5$  is automatic [10, Example 6.7]. Therefore, we obtain that there exists a pair  $(T, w_0), T \in \mathcal{T}$  for which the growth series  $\sum b_n z^n$  is not rational.  $\square$

Moreover, a Turing transducer of the class  $\mathcal{T}$  may have a function  $b_n, n = 0, \dots, \infty$  of intermediate growth.

*Example 5.* Miasnikov and Savchuk constructed an example of a 4-regular automatic graph which has intermediate growth [15]. Therefore, we obtain that there exists a pair  $(T, w_0), T \in \mathcal{T}$  for which the function  $b_n, n = 0, \dots, \infty$  has intermediate growth.  $\square$

We now consider the behavior of Følner function  $f_n, n = 1, \dots, \infty$  for Turing transducers of the class  $\mathcal{T}$ . Følner functions were first considered by A. Vershik for Cayley graphs of amenable groups [19]. Recall first some necessary definitions regarding Følner functions [7].

Let  $G$  be an amenable group with a finite set of generators  $Q \subseteq G$ . Put  $S = Q \cup Q^{-1}$ . Let  $E$  be the set of directed edges of  $\Gamma(G, S)$ . For a given finite set  $U \subseteq G$  the boundary  $\partial U$  is defined as

$$\partial U = \{u \in U \mid \exists v \in G[(u, v) \in E \wedge v \notin U]\}.$$

The function  $F\phi_{G,Q} : (0, 1) \rightarrow \mathbb{N}$  is defined as

$$F\phi_{G,Q}(\varepsilon) = \min\{\#U \mid \#\partial U < \varepsilon \#U\}.$$

The Følner function  $F\phi_{G,Q} : \mathbb{N} \rightarrow \mathbb{N}$  is defined as  $F\phi_{G,Q}(n) = F\phi_{G,Q}(\frac{1}{n})$ . The following claim is straightforward.

*Claim.* Suppose that the Cayley graph  $\Gamma(G, S)$  is presented by a Turing transducer  $T \in \mathcal{T}$ . Then for the Følner function  $f_n$  of  $T$ ,  $f_n = F\phi_{G,Q}(n)$ .  $\square$

In this subsection we say that  $f_1(n) \sim f_2(n)$  if there exists  $K \in \mathbb{N}$  such that  $f_1(Kn) \geq \frac{1}{K} f_2(n)$  and  $f_2(Kn) \geq \frac{1}{K} f_1(n)$ , i.e.,  $f_1(n)$  and  $f_2(n)$  are equivalent up to a quasi-isometry. Let  $Q' \subseteq G$  be another set generating  $G$ . Then  $F\phi_{G,Q}(n) \sim F\phi_{G,Q'}(n)$ . In this subsection Følner functions are considered up to quasi-isometries. So, instead of  $F\phi_{G,Q}(n)$ , we will write  $F\phi_G(n)$ .

Let  $G_1 = \mathbb{Z} \wr \mathbb{Z}$ . Put  $G_{i+1} = G_i \wr \mathbb{Z}, i \geq 1$ . It is shown [7, Example 3] that  $\text{Føl}_{G_i}(n) \sim n^{(n^i)}$ . It follows from [2, Theorem 3] that for every integer  $i \geq 1$  there exists a Turing transducer  $T_i \in \mathcal{T}$  for which a Cayley graph of  $G_i$  is presented by  $T_i$ . The following theorem shows that the logarithm of Følner functions for Turing transducers of the class  $\mathcal{T}$  can grow faster than any given polynomial.

**Theorem 6.** *For every integer  $i \geq 1$  there exists a Turing transducer of the class  $\mathcal{T}$  for which  $f_n \sim n^{(n^i)}$ .  $\square$*

*Remark 7.* Consider the group  $\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})$ . It is shown [7, Example 4] that  $\text{Føl}_{\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})}(n) \sim n^{(n^n)}$ . In particular,  $\text{Føl}_{\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})}(n)$  grows faster than  $\text{Føl}_{G_i}(n)$  for every  $i \geq 1$ . However, it is not known whether or not there exists a Turing transducer  $T \in \mathcal{T}$  for which a Cayley graph of  $\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})$  is presented by  $T$ .  $\square$

## 5.2 Random Walk and Average Length Growth Functions

Recall first some necessary definitions [20]. Let  $G$  be an infinite group with a set of generators  $Q = \{s_1, \dots, s_m\} \subseteq G$ . Put  $S = Q \cup Q^{-1} = \{s_1, \dots, s_m, s_1^{-1}, \dots, s_m^{-1}\}$ . For a given  $g \in G$  we denote by  $\ell_S(g)$  the minimal length of a word representing  $g$  in terms of  $S$ . We denote by  $B_n$  the ball of the radius  $n$ ,  $B_n = \{g \in G \mid \ell_S(g) \leq n\}$ . Let  $\mu$  be a symmetric measure defined on  $S$ , i.e.,  $\mu(s) = \mu(s^{-1})$  for all  $s \in S$ . The convolution  $\mu^{*n}(g)$  on  $B_n$  is defined as

$$\mu^{*n}(g) = \sum_{g=g_1 \dots g_n} \prod_{i=1, \dots, n} \mu(g_i),$$

where  $g_i \in S, i = 1, \dots, n$ .

Let  $c_n(g)$  be the number of words of length  $n$  over the alphabet  $S$  representing the element  $g \in G$ . If  $\mu$  is the uniform measure on  $S$ , then  $\mu^{*n}(g) = \frac{c_n(g)}{(2m)^n}$ . Therefore,  $\mu^{*n}(g)$  is the probability that a  $n$ -step simple symmetric random walk on the Cayley graph  $\Gamma(G, S)$ , which starts at the identity  $e \in G$ , ends up at the vertex  $g \in G$ . In this paper we consider only uniform measures  $\mu$ . We denote by  $E_{\mu^{*n}}[\ell_S]$  the average value of the functional  $\ell_S$  on the ball  $B_n$  with respect to the measure  $\mu^{*n}$ . For some Cayley graphs of wreath products of groups we will show asymptotic behavior of  $E_{\mu^{*n}}[\ell_S]$  of the form  $E_{\mu^{*n}}[\ell_S] \asymp f(n)$ , where  $g(n) \asymp f(n)$  means that  $\delta_1 f(n) \leq g(n) \leq \delta_2 f(n)$  for some constants  $\delta_2 \geq \delta_1 > 0$ .

Let  $T \in \mathcal{T}$  be a Turing transducer translating a language  $L$  into  $L'$ . Suppose that the Cayley graph  $\Gamma(G, S)$  is presented by  $T$ . Let us choose any word  $w_0 \in L$ . The Turing transducer  $T$  provides the bijection  $\psi : L \rightarrow G$  such that  $\psi^{-1}(e) = w_0$ . Therefore, we can consider the average of the functional  $|w|$  on the ball  $B_n$  with respect to the measure  $\mu^{*n}$ , where  $|w|$  is the length of a word  $w \in L$ . The following claim is straightforward.

*Claim.* For a  $n$ -step symmetric simple random walk on the Cayley graph  $\Gamma(G, S)$ ,  $E_{\mu^{*n}}[|w|] = \ell_n$ , where  $\ell_n$  is the  $n$ th element of the average length growth function of the pair  $(T, w_0)$ .  $\square$

The following proposition relates  $\ell_n$  and  $E_{\mu^{*n}}[\ell_S]$ .

**Proposition 8.** *There exist constants  $C_1$  and  $C_2$  such that  $\ell_n \leq C_1 E_{\mu^{*n}}[\ell_S] + C_2$  for all  $n$ .*

*Proof.* Recall that, by definition, there exists a constant  $c$  such that for every input  $x \in L$  and an output  $y_j \in L, j = 1, \dots, 2m, |y_j| \leq |x| + c$ . Put  $C_1 = c$  and  $C_2 = |w_0|$ . Therefore, we obtain that the inequality  $\ell_n \leq C_1 E_{\mu^{*n}}[\ell_S] + C_2$  holds for all  $n$ .  $\square$

It is easy to give examples of Turing transducers of the class  $\mathcal{T}$  for which  $\ell_n \asymp \sqrt{n}$  and the growth function  $b_n$  is polynomial using a unary-like representation of integers. See Example 9 below.

*Example 9.* Let  $Q = \{s_1, \dots, s_m\}$  be the standard set of generators of the group  $\mathbb{Z}^m$ , where  $s_i = (\delta_i^1, \dots, \delta_i^m)$  and  $\delta_i^j = 1$  if  $i = j$ ,  $\delta_i^j = 0$  if  $i \neq j$ . Put  $S = Q \cup Q^{-1}$ . It can be seen that there exists a  $(2m+1)$ -tape Turing transducer  $T \in \mathcal{T}$  translating a language  $L$  into a language  $L' \subseteq L^{2m}$  for which  $\Gamma(\mathbb{Z}^m, S)$  is presented by  $T$ . It is easy to see that a language  $L$  and an isomorphism between  $\Gamma_T$  and  $\Gamma(\mathbb{Z}^m, S)$  can be chosen in a way that  $\ell_S(g) = |w|$ , where  $g \in \mathbb{Z}^m$  and  $w \in L$  is the word corresponding to  $g$ . In particular, put the empty word  $\epsilon$  to be the representative of the identity  $(0, \dots, 0) \in \mathbb{Z}^m$ . Therefore, for such a Turing transducer  $T$ ,  $\ell_n = E_{\mu^{*n}}[\ell_S]$ . For a symmetric simple random walk on the  $m$ -dimensional grid,  $E_{\mu^{*n}}[\ell_S] \asymp \sqrt{n}$ . For the proof see, e.g., [17]. So, for the pair  $(T, \epsilon)$ ,  $\ell_n \asymp \sqrt{n}$ . The growth function  $b_n$  of  $(T, \epsilon)$  is polynomial. Thus, we obtain  $(2m+1)$ -tape Turing transducers  $T_m, m = 1, \dots, \infty$  for which  $\ell_n \asymp \sqrt{n}$  and the growth function  $b_n$  is polynomial.  $\square$

A more complicated technique is required in order to show an example of a Turing transducer of the class  $\mathcal{T}$  for which  $\ell_n \asymp \sqrt{n}$  and the growth function  $b_n$  is exponential. We will construct such a Turing transducer in Lemma 11.

Let  $H$  be a group with a set of generators  $S_H = \{t_1, \dots, t_k\}$ . Consider the group  $\mathbb{Z}_2 \wr H$ . Let  $h \in \mathbb{Z}_2^{(H)}$  be the function  $h : H \rightarrow \mathbb{Z}_2$  such that  $h(g) = e$  if  $g \neq e$  and  $h(e) = a$ , where  $a$  is the nontrivial element of  $\mathbb{Z}_2$ . Let  $Q = \{t, th, ht, hth | t \in S_H\}$  be the set of generators of the group  $\mathbb{Z}_2 \wr H$ . Put  $S = Q \cup Q^{-1}$ . Consider a symmetric simple random walk on the Cayley graph  $\Gamma(\mathbb{Z}_2 \wr H, S)$ . It is easy to see that a  $n$ -step random walk on  $\Gamma(\mathbb{Z}_2 \wr H, S)$  corresponds to a  $n$ -step random walk on  $H$ . Put  $P = S_H \cup S_H^{-1}$ . Let  $R_n$  be the number of different vertices visited after walking  $n$  steps on  $\Gamma(H, P)$ . We call  $R_n$  the range of a  $n$ -step random walk on  $\Gamma(H, P)$ . In the following proposition the asymptotic behavior of  $E_{\mu^{*n}}[\ell_S]$  is expressed in terms of  $E_{\mu^{*n}}[R_n]$  – the average range for a  $n$ -step random walk on  $\Gamma(H, P)$ .

**Proposition 10.** *Let  $H$  and  $S$  be as above. For a symmetric simple random walk on  $\Gamma(\mathbb{Z}_2 \wr H, S)$ ,  $E_{\mu^{*n}}[\ell_S] \asymp E_{\mu^{*n}}[R_n]$ .*

*Proof.* For the proof see [5, Lemma 2].  $\square$

**Lemma 11.** *There exists a set of generators  $S_1$  of the lamplighter group  $\mathbb{Z}_2 \wr \mathbb{Z}$  for which the following statements hold.*



- (a) For a simple symmetric random walk on  $\Gamma(\mathbb{Z}_2 \wr \mathbb{Z}, S_1)$ ,  $E_{\mu^{*n}}[\ell_{S_1}] \asymp \sqrt{n}$ .  
(b) There exists a Turing transducer  $T_1 \in \mathcal{T}$  such that  $\Gamma(\mathbb{Z}_2 \wr \mathbb{Z}, S_1)$  is presented by  $T_1$  and  $\ell_n \asymp \sqrt{n}$ .

*Proof.* Let us consider the lamplighter group  $\mathbb{Z}_2 \wr \mathbb{Z}$ . Let  $t$  be a generator of the subgroup  $\mathbb{Z} \leq \mathbb{Z}_2 \wr \mathbb{Z}$  and  $h \in \mathbb{Z}_2^{(\mathbb{Z})} \leq \mathbb{Z}_2 \wr \mathbb{Z}$  be the function  $h : \mathbb{Z} \rightarrow \mathbb{Z}_2$  such that  $h(z) = e$  if  $z \neq 0$  and  $h(0) = a$ . Let  $Q_1 = \{t, th, ht, hth\}$  be the set of generators of  $\mathbb{Z}_2 \wr \mathbb{Z}$  and  $S_1 = Q_1 \cup Q_1^{-1}$ . For a simple symmetric random walk on  $\Gamma(\mathbb{Z}, \{t, t^{-1}\})$ ,  $E_{\mu^{*n}}[R_n] \sim \sqrt{n}$ , where  $\sim$  here means asymptotic equivalence. For the proof see, e.g., [17]. Therefore, from Proposition 10 we obtain that for a simple symmetric random walk on  $\Gamma(\mathbb{Z}_2 \wr \mathbb{Z}, S_1)$ ,  $E_{\mu^{*n}}[\ell_{S_1}] \asymp \sqrt{n}$ .

Let  $Q'_1 = \{t, h\}$  be a set of generators of  $\mathbb{Z}_2 \wr \mathbb{Z}$ . Put  $S'_1 = Q'_1 \cup Q'^{-1}_1 = \{t, t^{-1}, h\}$ . In [2, Theorem 2] we constructed an automatic presentation of the Cayley graph  $\Gamma(\mathbb{Z}_2 \wr \mathbb{Z}, S'_1)$ , the bijection  $\psi_1 : L_1 \rightarrow \mathbb{Z}_2 \wr \mathbb{Z}$ , for which the inequalities  $\frac{1}{3}\ell_{S'_1}(g) + \frac{2}{3} \leq |w| \leq \ell_{S'_1}(g) + 1$  hold for all  $g \in \mathbb{Z}_2 \wr \mathbb{Z}$ , where  $L_1$  is a regular language,  $w = \psi_1^{-1}(g) \in L_1$  is the word corresponding to  $g$  and  $|w|$  is the length of  $w$ . It is easy to see that  $\frac{1}{2}\ell_{S_1}(g) \leq \ell_{S'_1}(g) \leq 3\ell_{S_1}(g)$ . Therefore, we obtain that  $\frac{1}{6}\ell_{S_1}(g) + \frac{2}{3} \leq |w| \leq 3\ell_{S_1}(g) + 1$  for all  $g \in \mathbb{Z}_2 \wr \mathbb{Z}$ . This implies that  $\frac{1}{6}E_{\mu^{*n}}[\ell_{S_1}] + \frac{2}{3} \leq E_{\mu^{*n}}[|w|] \leq 3E_{\mu^{*n}}[\ell_{S_1}] + 1$ . The bijection  $\psi_1 : L_1 \rightarrow \mathbb{Z}_2 \wr \mathbb{Z}$  provides an automatic presentation for the Cayley graph  $\Gamma(\mathbb{Z}_2 \wr \mathbb{Z}, S_1)$ . By Lemma 2, we obtain that there exists a 9-tape Turing transducer  $T_1 \in \mathcal{T}$  translating the language  $L_1$  into some language  $L'_1 \subseteq L_1^8$  for which  $\Gamma(\mathbb{Z}_2 \wr \mathbb{Z}, S_1)$  is presented by  $T_1$ . Therefore, we obtain that for  $T_1$ ,  $\ell_n \asymp \sqrt{n}$ . Since the growth function of the group  $\mathbb{Z}_2 \wr \mathbb{Z}$  is exponential, the growth function  $b_n$  of  $T_1$  is exponential.  $\square$

It is easy to give examples of Turing transducers of the class  $\mathcal{T}$  for which  $\ell_n \asymp n$  and the growth function  $b_n$  is exponential. See Example 12 below.

*Example 12.* Let  $F_m$  be the free group over  $m$  generators  $s_1, \dots, s_m$ . Put  $Q = \{s_1, \dots, s_m\}$  and  $S = Q \cup Q^{-1}$ . There exists a natural automatic presentation of the Cayley graph  $\Gamma(F_m, S)$ , the bijection  $\psi : L \rightarrow F_m$ , for which  $L$  is the language of all reduced words over the alphabet  $S$ . In particular, the empty word  $\epsilon$  represents the identity  $e \in F_m$ . The bijection  $\psi$  maps a word  $w \in L$  into the corresponding group element of  $F_m$ . It is clear that  $\ell_S(g) = |w|$ , where  $w = \psi^{-1}(g)$ . For a symmetric simple random walk on  $\Gamma(F_m, S)$ ,  $E_{\mu^{*n}}[\ell_S] \asymp n$ . Therefore,  $E_{\mu^{*n}}[|w|] \asymp n$ . Therefore, for each  $m > 1$  we obtain the pair  $(T, \epsilon)$ ,  $T \in \mathcal{T}$  for which  $\ell_n \asymp n$ . Since the growth function of the free group  $F_m$  is exponential, the growth function  $b_n$  of the pair  $(T, \epsilon)$  is exponential.  $\square$

Is there a Turing transducer of the class  $\mathcal{T}$  for which  $\ell_n$  grows between  $\sqrt{n}$  and  $n$ ? We will answer on this question positively in Theorem 14 which follows from Proposition 13 below.

Let  $G$  be a group with a set of generators  $S_G = \{g_1, \dots, g_m\}$ . Put  $P = S_G \cup S_G^{-1}$ . Assume that for a symmetric simple random walk on  $\Gamma(G, P)$ ,  $\ell_n(\mu) \asymp n^\alpha$  for some  $0 < \alpha \leq 1$ . Consider the wreath product  $G \wr \mathbb{Z}$ . Let  $t$  be a generator of the subgroup  $\mathbb{Z} \leq G \wr \mathbb{Z}$ . Let  $h_i \in G^{(\mathbb{Z})} \leq G \wr \mathbb{Z}$ ,  $i = 1, \dots, m$  be the functions  $h_i : \mathbb{Z} \rightarrow G$  such that  $h_i(z) = e$  if  $z \neq 0$  and  $h_i(0) = g_i$ . Put  $Q = \{h_i^p t h_j^q \mid i, j =$

$1, \dots, m; p, q = -1, 0, 1\}$  to be the set of generators of the group  $G \wr \mathbb{Z}$  and  $S = Q \cup Q^{-1}$ . Consider a  $n$ -step random walk on  $\Gamma(G \wr \mathbb{Z}, S)$ . The following proposition shows asymptotic behavior of  $E_{\mu^{*n}}[\ell_S]$ .

**Proposition 13.** *Let  $G$ ,  $S$  and  $\alpha$  be as above. For a symmetric simple random walk on  $\Gamma(G \wr \mathbb{Z}, S)$ ,  $E_{\mu^{*n}}[\ell_S] \asymp n^{\frac{1+\alpha}{2}}$ .*

*Proof.* For the proof see [8, Lemma 3].  $\square$

**Theorem 14.** *For every  $\alpha < 1$  there exists a Turing transducer  $T \in \mathcal{T}$  for which  $\ell_n \asymp n^\beta$  for some  $\beta$  such that  $\alpha < \beta < 1$  and the growth function  $b_n$  is exponential.*

*Proof.* Let us consider the sequence of wreath products  $G_m, m = 1, \dots, \infty$  such that  $G_1 = \mathbb{Z}_2 \wr \mathbb{Z}$  and  $G_{m+1} = G_m \wr \mathbb{Z}$ ,  $m \geq 1$ . From Lemma 11 (a) and Proposition 13 we obtain that for every  $m > 1$  there exists a proper set of generators  $Q_m \subseteq G_m$  such that for a symmetric simple random walk on the Cayley graph  $\Gamma(G_m, S_m)$ ,  $E_{\mu^{*n}}[\ell_{S_m}] \asymp n^{1-\frac{1}{2^m}}$ , where  $S_m = Q_m \cup Q_m^{-1}$ . It follows from [2, Theorem 3] that for every  $m > 1$  there is an automatic presentation of the Cayley graph  $\Gamma(G_m, S'_m)$ , the bijection  $\psi_m : L_m \rightarrow G_m$ , for which the inequalities  $\delta'_1 \ell_{S'_m}(g) + \lambda'_1 \leq |w| \leq \delta'_2 \ell_{S'_m}(g) + \lambda'_2$  hold for all  $g \in G_m$  for some constants  $\delta'_2 > \delta'_1 > 0, \lambda'_1, \lambda'_2$ , where  $L_m$  is a regular language and  $S'_m = Q'_m \cup Q'^{-1}_m$  for some proper set of generators  $Q'_m \subseteq G_m$ , and  $w = \psi_m^{-1}(g)$  is the word representing  $g$ . Therefore, the inequalities  $\delta_1 \ell_{S_m}(g) + \lambda_1 \leq |w| \leq \delta_2 \ell_{S_m}(g) + \lambda_2$  hold for all  $g \in G_m$  for some constants  $\delta_2 > \delta_1 > 0, \lambda_1, \lambda_2$ . This implies that  $\delta_1 E_{\mu^{*n}}[\ell_{S_m}] + \lambda_1 \leq E_{\mu^{*n}}[|w|] \leq \delta_2 E_{\mu^{*n}}[\ell_{S_m}] + \lambda_2$ . Therefore,  $E_{\mu^{*n}}[|w|] \asymp n^{1-\frac{1}{2^m}}$ .

For every  $m > 1$  the bijection  $\psi_m : L_m \rightarrow G_m$  provides an automatic presentation of the Cayley graph  $\Gamma(G_m, S_m)$ . It follows from Lemma 2 that there is a  $(k_m + 1)$ -tape Turing transducer  $T_m \in \mathcal{T}$  translating the language  $L_m$  into  $L'_m \subseteq L_m^{k_m}$  for which, after proper relabeling,  $\Gamma_{T_m} \cong \Gamma(G_m, S_m)$ . The numbers  $k_m, m = 1, \dots, \infty$  can be obtained recurrently as follows. It is easy to see that  $k_{m+1} = 2(k_m + 1)^2$  for  $m \geq 1$  and  $k_1 = 8$ , which is simply the number of elements in  $S_1$  (see Lemma 11). So, we obtain that for  $T_m, m > 1$ ,  $\ell_n \asymp n^{1-\frac{1}{2^m}}$ . For every  $m > 1$ , since the growth function of the group  $G_m$  is exponential, the growth function  $b_n$  of  $T_m$  is exponential.  $\square$

## 6 Discussion

In this paper we addressed the problem of finding characterizations of Cayley automatic groups. Our approach was to define and then study three numerical characteristics of Turing transducers of the special class  $\mathcal{T}$ . This class of Turing transducers was obtained from automatic presentations of labeled directed graphs. The numerical characteristics that we defined are the analogs of growth functions, Følner functions and drifts of simple random walks for Cayley graphs of groups. We hope that further study of asymptotic behavior of these three

numerical characteristics of Turing transducers of the class  $\mathcal{T}$  will yield some characterizations for Cayley automatic groups.

Two open questions are apparent from the results of Section 5.

- Theorem 6 shows that for every integer  $i \geq 1$  there exists a Turing transducer of the class  $\mathcal{T}$  for which  $f_n \sim n^{(n^i)}$ . Is there a Turing transducer  $T \in \mathcal{T}$  for which the Følner function grows faster than  $n^{(n^i)}$  for all  $i \geq 1$ ?
- Theorem 14 tells us that for every  $\alpha < 1$  there exists a Turing transducer  $T \in \mathcal{T}$  for which  $\ell_n \asymp n^\beta$  for some  $\beta$  such that  $\alpha < \beta < 1$ . Is there a Turing transducer  $T \in \mathcal{T}$  for which  $\ell_n$  grows faster than  $n^\alpha$  for every  $\alpha < 1$  but slower than  $n$ ?

## Acknowledgments

The author thanks Bakhadyr Khoussainov and the anonymous reviewers for useful suggestions. The author thanks Sunny Daniels for proofreading a draft of this paper and making several changes to it.

## References

1. Berdinsky, D., Khoussainov, B.: On automatic transitive graphs. In: Shur, A., Volkov, M. (eds.) *Developments in Language Theory 2014, Lecture Notes in Computer Science*, vol. 8633, pp. 1–12. Springer Berlin Heidelberg (2014)
2. Berdinsky, D., Khoussainov, B.: Cayley automatic representations of wreath products. *International Journal of Foundations of Computer Science* 27(2), 147 – 159 (2016)
3. Calude, C., Calude, E., Khoussainov, B.: Deterministic automata simulation, universality and minimality. *Annals of Pure and Applied Logic* 90(1), 263–276 (1997)
4. Calude, C.S., Calude, E., Khoussainov, B.: Finite nondeterministic automata: Simulation and minimality. *Theoretical Computer Science* 242(1), 219–235 (2000)
5. Dyubina, A.: An example of the rate of growth for a random walk on a group. *Russian Mathematical Surveys* 54(5), 1023–1024 (1999)
6. Epstein, D.B.A., Cannon, J.W., Holt, D.F., Levy, S.V.F., Paterson, M.S., Thurston, W.P.: *Word Processing in Groups*. Jones and Barlett Publishers. Boston, MA (1992)
7. Erschler, A.: On Isoperimetric Profiles of Finitely Generated Groups. *Geometriae Dedicata* 100(1), 157–171 (2003)
8. Erschler, A.: On the asymptotics of drift. *Journal of Mathematical Sciences* 121(3), 2437–2440 (2004)
9. Kargapolov, M.I., Merzljakov, J.I.: *Fundamentals of the theory of groups*. Springer-Verlag New York Inc. (1979)
10. Kharlampovich, O., Khoussainov, B., Miasnikov, A.: From automatic structures to automatic groups. *Groups, Geometry, and Dynamics* 8(1), 157–198 (2014)
11. Khoussainov, B., Minnes, M.: Three lectures on automatic structures. *Proceedings of Logic Colloquium* pp. 132–176 (2007)

12. Khoussainov, B., Nerode, A.: Automatic presentations of structures. In: Leivant, D. (ed.) *Logic and Computational Complexity*, Lecture Notes in Computer Science, vol. 960, pp. 367–392. Springer Berlin Heidelberg (1995)
13. Khoussainov, B., Nerode, A.: Open questions in the theory of automatic structures. *Bulletin of the EATCS* 94, 181–204 (2008)
14. Meduna, A.: *Automata and Languages. Theory and Applications*. Springer-Verlag London Ltd. (2000)
15. Miasnikov, A., Savchuk, D.: An example of an automatic graph of intermediate growth. *Ann. Pure Appl. Logic* 166(10), 1037–1048 (2015)
16. Oliver, G.P., Thomas, R.M.: Automatic presentations for finitely generated groups. In: Diekert, V., Durand, B. (eds.) *STACS 2005*, Lecture Notes in Computer Science, vol. 3404, pp. 693–704. Springer Berlin Heidelberg (2005)
17. Spitzer, F.: *Principles of random walk*. Van Nostrand, Princeton (1964)
18. Stoll, M.: Rational and transcendental growth series for the higher Heisenberg groups. *Invent. Math.* 126, 85–109 (1996)
19. Vershik, A.: Countable groups that are close to finite ones, Appendix in F. P. Greenleaf, *Invariant Means on Topological Groups and their Applications*, Moscow, Mir, 1973 (in Russian), a revised English translation: Amenability and approximation of infinite groups. *Selecta Math.* 2(4), 311–330 (1982)
20. Vershik, A.: Numerical characteristics of groups and corresponding relations. *Journal of Mathematical Sciences* 107(5), 4147–4156 (2001)